The Twelvefold Way

Ben Wallberg

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Abstract

In combinatorics, the twelvefold way is a systematic classification of 12 related enumerative problems concerning two finite sets, which include the classical problems of counting permutations, combinations, multisets, and partitions either of a set or of a number. This is my cheat sheet, compiled from Stanley [3] by way of Knuth [1] and Wikipedia [4], et al. I use this aid to help me understand and solve Project Euler [2] problems. Project Euler is a series of challenging mathematical/computer programming problems that require more than just mathematical insights to solve.

***f*-class**		**Any** f	**Injective** f	**Surjective** f
	balls per urn	**unrestricted**	$\leq$ **1**	$\geq$ **1**
f		n-sequence in $\mathbf{X}$	n-permutation of $\mathbf{X}$	composition of $\mathbf{N}$ with x subsets
	n labeled balls, m labeled urns	n-tuples of m things x^n	n-permutation of m things $x^{\underline{n}}$	partitions of $\{1, \ldots, n\}$ into m ordered parts $x!\left\{ n \atop x \right\}$
$f \circ \mathbf{S}_n$		n-multisubset of $\mathbf{X}$	n-subset of $\mathbf{X}$	composition of n with x terms
	n unlabeled balls, m labeled urns	n-multicombinations of m things $\left(\!\!\binom{x}{n}\!\!\right)$	n-combinations of m things $\binom{x}{n}$	compositions of n into m parts $\left(\!\!\binom{x}{n-x}\!\!\right)$
$\mathbf{S}_x \circ f$		partition of $\mathbf{N}$ into $\leq x$ subsets	partition of $\mathbf{N}$ into $\leq x$ elements	partition of $\mathbf{N}$ into x subsets
	n labeled balls, m unlabeled urns	partitions of $\{1, \ldots, n\}$ into $\leq m$ parts $\sum_{k=0}^{x} \left\{ n \atop x \right\}$	n pigeons into m holes $[n \leq x]$	partitions of $\{1, \ldots, n\}$ into m parts $\left\{ n \atop x \right\}$
$\mathbf{S}_x \circ f \circ \mathbf{S}_n$		partition of n into $\leq x$ parts	partition of n into $\leq x$ parts 1	partition of n into x parts
	n unlabeled balls, m unlabeled urns	partitions of n into $\leq m$ parts $p_x(n + x)$	n pigeons into m holes $[n \leq x]$	partitions of n into m parts $p_x(n)$

Equations of Interest

Falling Factorial or the first n elements of $x!$

$$x^{\underline{n}} = \frac{x!}{(x-n)!} = x(x-1)(x-2)\cdots(x-n+1)$$

$$x^{\underline{x}} = \frac{x!}{(x-x)!} = x!$$

Binomial Coefficient (aka n choose k)

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n^{\underline{k}}}{k!}$$

The recurrence relation for Pascal's Triangle:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Multiset (aka bag or multicombination) is modification of the concept of a set that allows for multiple instances of its elements

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k} = \frac{n^{\underline{k}}}{k!}$$

$$\left(\!\!\binom{x}{n-x}\!\!\right) = \binom{n-1}{n-x}$$

Stirling number of the second kind (or Stirling partition number) is the number of ways to partition a set of n objects into k non-empty subsets

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} = x^n$$

Bell number is the total number of partitions of a set with n members over all values of k

$$B_n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}$$

Details

n-sequence in X (f, **Any** f)

x^n

`itertools.product(range(x), repeat=n)`

n-permutation of X (f, **Injective** f)

$x^{\underline{n}}$

`itertools.permutations(range(x), n)`

composition of N with x subsets (f, **Surjective** f)

$x!\left\{ {n \atop x} \right\}$

n-multisubset of X ($f \circ \mathbf{S}_n$, **Any** f)

$\left(\!\!\binom{x}{n}\!\!\right)$

n-subset of X $(f \circ \mathbf{S}_n$, **Injective** $f)$

$\binom{x}{n}$

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itertools.combinations(range(x), n)
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composition of n with x terms $(f \circ \mathbf{S}_n$, **Surjective** $f)$

$$\left(\!\!\binom{x}{n-x}\!\!\right)$$

partition of N into $\leq x$ subsets $(\mathbf{S}_x \circ f$, **Any** $f)$

$\sum_{k=0}^{x} \left\{ {n \atop x} \right\}$

partition of N into $\leq x$ elements $(\mathbf{S}_x \circ f$, **Injective** $f)$

$[n \leq x]$

partition of N into x subsets $(\mathbf{S}_x \circ f$, **Surjective** $f)$

$\left\{ {n \atop x} \right\}$

partition of n into $\leq x$ parts $(\mathbf{S}_x \circ f \circ \mathbf{S}_n$, **Any** $f)$

$p_x(n + x)$

partition of n into $\leq x$ parts 1 $(\mathbf{S}_x \circ f \circ \mathbf{S}_n$, **Injective** $f)$

$[n \leq x]$

partition of n into x parts $(\mathbf{S}_x \circ f \circ \mathbf{S}_n$, **Surjective** $f)$

$p_x(n)$

References

[1] Donald E. Knuth. *The Art of Computer Programming: Volume 4A, Combinatorial Algorithms, Part 1*. English. Upper Saddle River, NJ: Addison-Wesley, 2011. ISBN: 978-0-201-03804-0.

[2] *Project Euler*. URL: `https://projecteuler.net/`.

[3] Richard P. Stanley. *Enumerative Combinatorics, Volume 1*. English. 2nd ed. Vol. 1. Cambridge studies in advanced mathematics 49. Cambridge ; Cambridge University Press, 1997. ISBN: 0-521-55309-1.

[4] *Twelvefold way - Wikipedia*. URL: `https://en.wikipedia.org/wiki/Twelvefold_way`.